

Zeros distribution of gaussian entire functions

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Abstract

In this paper we consider a random entire function of the form $f(z, \omega) = \sum_{n=0}^{+\infty} \xi_n(\omega) a_n z^n$, where $\xi_n(\omega)$ are independent standard complex gaussian random variables and $a_n \in \mathbb{C}$ satisfy the relations $\overline{\lim}_{n \rightarrow +\infty} \sqrt[n]{|a_n|} = 0$ and $\#\{n: a_n \neq 0\} = +\infty$. We investigate asymptotic properties of the probability $P_0(r) = P\{\omega: f(z, \omega) \text{ has no zeros inside } r\mathbb{D}\}$. Denote $p_0(r) = \ln^- P_0(r)$, $N(r) = \#\{n: \ln(|a_n| r^n) > 0\}$, $s(r) = \sum_{n=0}^{+\infty} \ln^+(|a_n| r^n)$. Assuming that $a_0 \neq 0$ we prove that

$$0 \leq \lim_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln(p_0(r) - s(r))}{\ln s(r)}, \quad \overline{\lim}_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln(p_0(r) - s(r))}{\ln s(r)} \leq \frac{1}{2},$$

$$\lim_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln(p_0(r) - s(r))}{\ln N(r)} = 1.$$

where E is a set of finite logarithmic measure. Remark that the previous inequalities are sharp. Also we give an answer to open question from [25, p. 119].

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1 Introduction

One of the problems on random functions is investigation of value distribution of there functions and also asymptotic properties of the probability of absence

of zeros in a disc (“hole probability”). These problems were considered in papers of J. E. Littlewood and A. C. Offord ([1]–[6]), M. Sodin and B. Tsirelson ([7]–[9]), Yu. Peres and B. Virag ([10], [11]), P. V. Filevych and M. P. Mahola ([12]–[14]), M. Sodin ([15]–[17]), F. Nazarov, M. Sodin and A. Volberg ([18], [19]), M. Krishnapur ([20]), A. Nishry ([21]–[25]) and many others.

So, in [9] it was considered a random entire function of the form

$$\psi(z, \omega) = \sum_{k=0}^{+\infty} \xi_k(\omega) \frac{z^k}{\sqrt{k!}}, \quad (1)$$

where $\{\xi_k(\omega)\}$ are independent complex valued random variables with the density function

$$p_{\xi_k}(z) = \frac{1}{\pi} e^{-|z|^2}, \quad z \in \mathbb{C}, \quad k \in \mathbb{Z}_+.$$

We denote such a distribution by $\mathcal{N}_{\mathbb{C}}(0, 1)$.

Let us denote by $n_\psi(r, \omega)$ the counting function of zeros of the function $\psi(z, \omega)$ in $r\mathbb{D} = \{z: |z| < r\}$. Then ([9]) for any $\delta \in (0, 1/4]$ and all $r \geq 1$ we have

$$P\left\{\omega: \left|\frac{n(r, \omega)}{r^2} - 1\right| \geq \delta\right\} \leq \exp(-c(\delta)r^4),$$

where the constant $c(\delta)$ depends only on δ . Also in [9] it was investigated the probability of absence of zeros of the function $\psi(z, \omega)$,

$$P_0(r) = P\{\omega: \psi(z, \omega) \neq 0 \text{ inside } r\mathbb{D}\}.$$

In particular, it was proved in [9] that there exist constants $c_1, c_2 > 0$ such that

$$\exp(-c_1 r^4) \leq P_0(r) \leq \exp(-c_2 r^4) \quad (r \geq 1).$$

Also in [9] the authors put the following question: *Does there exist the limit*

$$\lim_{r \rightarrow +\infty} \frac{\ln^- P_0(r)}{r^4} ?$$

We find the answer to this question in [21]. For the function $\psi(z, \omega)$ it was proved that

$$\lim_{r \rightarrow +\infty} \frac{\ln^- P_0(r)}{r^4} = \frac{3e^2}{4}.$$

In [21] it was proved that if all of $\xi_n(\omega): \xi_n(\Omega) \subset K$, where $K \subset \mathbb{C}$ and $0 \notin K$ then there exists $r_0(K) < +\infty$ such that $\psi(z, \omega)$ must vanish somewhere in the disc $r_0\mathbb{D}$.

For the function of the form (1) one can fix the disc of radius r and ask for the asymptotic behaviour of $P\{\omega: n_\psi(r, \omega) \geq m\}$ as $m \rightarrow +\infty$. So in [20] it was proved, that for any $r > 0$ we get

$$\ln P\{\omega: n_\psi(r, \omega) \geq m\} = -\frac{1}{2}m^2 \ln m(1 + o(1)) \quad (m \rightarrow +\infty).$$

Very large deviations of zeros of function (1) were also considered in [19]. There we find such a relation

$$\lim_{r \rightarrow +\infty} \frac{\ln \ln \left(\frac{1}{P\{\omega: |n_\psi(r, \omega) - r^2| > r^\alpha\}} \right)}{\ln r} = \begin{cases} 2\alpha - 1, & \frac{1}{2} \leq \alpha \leq 1; \\ 3\alpha - 2, & 1 \leq \alpha \leq 2; \\ 2\alpha, & \alpha \geq 2. \end{cases}$$

More generally, in [24, 25] it was considered gaussian entire functions of the form

$$f(z, \omega) = \sum_{n=0}^{+\infty} \xi_k(\omega) a_n z^n,$$

where $a_0 \neq 0$, $n \in \mathbb{Z}_+$, $\overline{\lim}_{n \rightarrow +\infty} \sqrt[n]{|a_n|} = 0$. If $\varepsilon > 0$, then there exists ([24, 25]) a set of finite logarithmic measure $E \subset (1, +\infty)$ ($\int_E \frac{dr}{r} < +\infty$) such that for all $r \in (1, +\infty) \setminus E$ we obtain

$$s(r) - s^{1/2+\varepsilon}(r) \leq p_0(r) \leq s(r) + s^{1/2+\varepsilon}(r), \quad s(r) = \sum_{n=0}^{+\infty} \ln^+(a_n r^n). \quad (2)$$

One can find similar results for gaussian analytic functions in the unit disc in [20], [10], [11], [17].

Also in [25, p. 119], it was formulated the following question: *Is the error term in inequality (2) optimal for a regular sequence of the coefficients $\{a_n\}$?* The **aim** of this paper is to answer this question.

2 Notation

In this section we consider the functions of the form

$$f(z, \omega) = \sum_{n=0}^{+\infty} \xi_n(\omega) a_n z^n, \quad (3)$$

where $\xi_n(\omega) \in \mathcal{N}_{\mathbb{C}}(0, 1)$ and $a_n \in \mathbb{C}$, $n \in \mathbb{Z}_+$ such that $\#\{n: a_n \neq 0\} = +\infty$, $\overline{\lim}_{n \rightarrow +\infty} \sqrt[n]{|a_n|} = 0$.

In this paper we study asymptotic behaviour of

$$p_0(r) = \ln^- P\{\omega: n_f(r, \omega) = 0\}$$

as $r \rightarrow +\infty$ for random entire functions of the form (3).

For $r > 0$, $\delta > 0$ denote

$$\begin{aligned}\mathcal{N}' &= \{n: a_n = 0\}, \quad \mathcal{N}(r) = \{n: \ln(|a_n|r^n) > 0\}, \\ \mathcal{N}_\delta(r) &= \{n: \ln(|a_n|r^n) > -\delta n\}, \quad N(r) = \#\mathcal{N}(r), \quad N_\delta(r) = \#\mathcal{N}_\delta(r), \\ m(r) &= \sum_{n \in \mathcal{N}(r)} n, \quad m_\delta(r) = \sum_{n \in \mathcal{N}_\delta(r)} n, \quad s(r) = 2 \sum_{n \in \mathcal{N}(r)} \ln(|a_n|r^n), \\ \mu_f(r) &= \max\{|a_n|r^n: n \in \mathbb{Z}_+\}, \quad \nu_f(r) = \max\{n: \mu_f(r) = |a_n|r^n\}, \\ M_f(r) &= \max\{|f(z)|: |z| \leq r\}, \quad S_f^2(r) = \sum_{n=0}^{+\infty} |a_n|^2 r^{2n}.\end{aligned}$$

3 Auxiliary lemmas

Lemma 3.1 (Borel-Nevanlinna, [27]). *Let $u(r)$ be a nondecreasing continuous function on $[r_0; +\infty)$ and $\lim_{r \rightarrow +\infty} u(r) = +\infty$, and $\varphi(u)$ be a continuous nonincreasing positive function defined on $[u_0; +\infty)$ and*

- 1) $u_0 = u(r_0)$;
- 2) $\lim_{u \rightarrow +\infty} \varphi(u) = 0$;
- 3) $\int_{u_0}^{+\infty} \varphi(u) du < +\infty$.

Then for all $r \geq r_0$ outside a set E of finite measure we have

$$u\{r - \varphi(u(r))\} > u(r) - 1.$$

We need the following elementary corollary of this lemma.

Lemma 3.2. *There exists a set $E \subset (1; +\infty)$ of finite logarithmic measure such that for all $r \in (1; +\infty) \setminus E$ we obtain*

$$m(re^{-\delta}) > em(r) \exp\{-2\sqrt{\ln m(r)}\}, \quad m(re^\delta) < em(r) \exp\{2\sqrt{\ln m(r)}\},$$

where $\delta = \frac{1}{2\ln m(r)}$.

Lemma 3.3. *Let $\varepsilon > 0$. There exists a set $E \subset (1; +\infty)$ of finite logarithmic measure such that for all $r \in (1; +\infty) \setminus E$ we have*

$$N(r) < s^{1/2}(r) \exp\{(1 + \varepsilon)\sqrt{\ln s(r)}\}. \quad (4)$$

Proof. Remark that (see also [24])

$$N_{-\delta}(r) = \#\{n: |a_n|r^n \geq e^{\delta n}\} = \#\{n: |a_n|(re^{-\delta})^n \geq 1\} = N(re^{-\delta})$$

and

$$m(r) = \sum_{n \in \mathcal{N}(r)} n \geq \sum_{n=0}^{N(r)-1} n = \frac{(N(r)-1)N(r)}{2} > \frac{N^2(r)}{e}$$

for $r > r_0$, where $N(r_0) > 4$. So we obtain for $r \in (1; +\infty) \setminus E$

$$\begin{aligned} \frac{s(r)}{2} &= \sum_{n \in \mathcal{N}(r)} \ln(|a_n|r^n) \geq \sum_{n \in \mathcal{N}_{-\delta}(r)} \ln(|a_n|r^n) \geq \sum_{n \in \mathcal{N}_{-\delta}(r)} n\delta = \\ &= \delta m(re^{-\delta}) > \frac{e}{2 \ln m(r)} m(r) \exp\{-2\sqrt{\ln m(r)}\}. \end{aligned}$$

Then

$$\ln s(r) > 1 + \ln m(r) - 2\sqrt{\ln m(r)} - \ln \ln m(r)$$

and for $r > r_2$ we get $\ln m(r) < 2 \ln s(r)$. So for any $\varepsilon > 0$

$$\begin{aligned} s(r) &> em(r) \exp\{-2\sqrt{\ln m(r)} - \ln \ln m(r)\} > \\ &> e \frac{N^2(r)}{e} \exp\{-2\sqrt{(1+\varepsilon) \ln s(r) - \ln((1+\varepsilon) \ln s(r))}\} > \\ &> N^2(r) \exp\{-(2+2\varepsilon)\sqrt{\ln s(r)}\}, \end{aligned}$$

as $r \rightarrow +\infty$ outside some set of finite logarithmic measure. \square

Let us note that the exponent $1/2$ in the inequality (4) can not be replaced by a smaller number.

Lemma 3.4. *There exist a random entire function of the form (3) and a set $E \subset (1; +\infty)$ of finite logarithmic measure such that for all $r \in (1; +\infty) \setminus E$ we have*

$$N(r) > \frac{\sqrt{s(r)}}{\ln^3 s(r)}.$$

Proof. Choose

$$f(z) = 1 + \sum_{n=1}^{+\infty} \frac{z^n}{\left(\frac{n}{2}\right)^{\frac{n}{2}}}.$$

Then the function $y(n) = \ln a_n = -\frac{n}{2} \ln\left(\frac{n}{2}\right)$ is concave function and the sequence $\{a_n\}$ is log-concave([22], [26]). Also by Stirling's formula we obtain

$$\begin{aligned} M_f(r) &= 1 + \sum_{n=1}^{+\infty} \frac{r^n}{\left(\frac{n}{2}\right)^{\frac{n}{2}}} > 1 + \sum_{n/2=1}^{+\infty} \frac{r^n}{\left(\frac{n}{2}\right)^{\frac{n}{2}}} = 1 + \sum_{m=1}^{+\infty} \frac{r^{2m}}{m^m} > 1 + \sum_{m=1}^{+\infty} \frac{r^{2m}}{m!e^m} = \\ &= \exp\left\{\frac{r^2}{e}\right\}, \quad \ln M_f(r) > \frac{r^2}{e}. \end{aligned}$$

By Wiman-Valiron's theorem there exists a set E of finite logarithmic measure such that for all $r \in (1; +\infty) \setminus E$ we get $\ln \mu_f(r) + \ln \ln \mu_f(r) > \ln M_f(r) > r^2/e$, $\ln \mu_f(r) > r^2/2e$ and finally

$$\frac{r^2}{2e} < \ln \mu_f(r) = \ln a_\nu + \nu_f(r) \ln r, \quad \nu_f(r) > \frac{1}{\ln r} \left(\frac{r^2}{2e} - \ln a_\nu \right) > r, \quad r \rightarrow +\infty.$$

Therefore, outside a set E of finite logarithmic measure we get ([22])

$$\begin{aligned} s(r) &< 2(N(r) + 1) \ln \mu_f(r) < \ln^2 \mu_f(r) (\ln \ln \mu_f(r))^2 = \\ &= \ln^3 r \frac{\ln^2 \mu_f(r)}{\ln^2 r} \frac{(\ln \ln \mu_f(r))^2}{\ln r} < \ln^3 r \nu_f^2(r) \ln^2 \nu_f(r) < \\ &< \nu_f^2(r) \ln^5 \nu_f(r) < N^2(r) \ln^5 N(r) < N^2(r) \ln^5 s(r), \\ N(r) &> \sqrt{\frac{s(r)}{\ln^5 s(r)}} > \frac{\sqrt{s(r)}}{\ln^3 s(r)}. \end{aligned}$$

□

Also we will use the following lemma.

Lemma 3.5. *Let $\{\zeta_n(\omega)\}$ be a sequence of independent identically distributed random variables, such that $M|\zeta_n| < +\infty$ and $M(\frac{1}{|\zeta_n|}) < +\infty$, $n \in \mathbb{Z}_+$. Then*

$$P\left\{\omega: (\exists N^*(\omega))(\forall n > N^*(\omega)) \left[\frac{1}{n} \leq |\zeta_n(\omega)| \leq n \right]\right\} = 1.$$

Proof. Let $F_{|\zeta|}(t) = F_{|\zeta_n|}(t)$ be the distribution function of the random variable $|\zeta_n|$, $n \in \mathbb{Z}_+$.

Denote $B_m = \{\omega: |\zeta_m(\omega)| \geq m\}$, $m \in \mathbb{Z}_+$. Then

$$\begin{aligned} \sum_{m=1}^{+\infty} P\{\omega: |\zeta_m(\omega)| \geq m\} &= \sum_{m=1}^{+\infty} \int_{|t| \geq m} dF_{|\zeta|}(t) = \sum_{m=1}^{+\infty} \sum_{s=m}^{+\infty} \int_{|t| \in [s, s+1)} dF_{|\zeta|}(t) = \\ &= \sum_{s=1}^{+\infty} \sum_{m=1}^s \int_{|t| \in [s, s+1)} dF_{|\zeta|}(t) = \sum_{s=1}^{+\infty} s \int_{|t| \in [s, s+1)} dF_{|\zeta|}(t) \leq \\ &\leq \sum_{s=1}^{+\infty} \int_{|t| \in [s, s+1)} |t| dF_{|\zeta|}(t) = M|\zeta| < +\infty. \end{aligned}$$

Therefore we obtain $\sum_{m=1}^{+\infty} P(B_m) < +\infty$. So, by the Borel-Cantelli lemma only finite quantity of the events B_n may occur. Then

$$P(A_1) = P\left\{\omega: (\exists N_1^*(\omega))(\forall n > N_1^*(\omega)) \left[|\zeta_n(\omega)| \leq n \right]\right\} = 1.$$

Since $M(\frac{1}{|\zeta|}) < +\infty$, we get similarly for random variable $\frac{1}{|\zeta(\omega)|}$

$$\begin{aligned} P(A_2) &= P\left\{\omega: (\exists N_2^*(\omega))(\forall n > N_2^*(\omega)) \left[\frac{1}{|\zeta_n(\omega)|} \leq n \right]\right\} = \\ &= P\left\{\omega: (\exists N_2^*(\omega))(\forall n > N_2^*(\omega)) \left[|\zeta_n(\omega)| \geq \frac{1}{n} \right]\right\} = 1. \end{aligned}$$

Finally,

$$P(A_1 \cap A_2) = P\left\{\omega: (\exists N^*(\omega))(\forall n > N^*(\omega)) \left[\frac{1}{n} \leq |\zeta_n(\omega)| \leq n \right]\right\} = 1,$$

where $N^*(\omega) = \max\{N_1^*(\omega), N_2^*(\omega)\}$. \square

The random variables $\xi_n \in \mathcal{N}_{\mathbb{C}}(0, 1)$, $n \in \mathbb{Z}_+$ satisfy conditions of Lemma 3.5. So, we get the following result.

Lemma 3.6. *Let $\xi_n \in \mathcal{N}_{\mathbb{C}}(0, 1)$, $n \in \mathbb{Z}_+$. Then*

$$P\left\{\omega: (\exists N^*(\omega))(\forall n > N^*(\omega)) \left[\frac{1}{n} \leq |\xi_n(\omega)| \leq n \right]\right\} = 1.$$

4 Upper and lower bounds for $p_0(r)$

Theorem 4.1. *Let $\varepsilon > 0$ and $f(z, \omega)$ be random entire function of the form (3) with $a_0 \neq 0$. There exists a set $E \subset (1; +\infty)$ of finite logarithmic measure such that for all $r \in (1; +\infty) \setminus E$ we have*

$$p_0(r) \leq s(r) + N(r) \exp\{(2 + \varepsilon)\sqrt{\ln N(r)}\}. \quad (5)$$

Proof. Similarly as in [24], we consider the event $\Omega_1 = \cap_{i=1}^4 A_i$, where

$$\begin{aligned} A_1 &= \left\{\omega: |\xi_0(\omega)| \geq \frac{2eN^{1/3}(r) \exp\{2\sqrt{\ln N(r)}\}}{|a_0|}\right\}, \\ A_2 &= \left\{\omega: (\forall n \in \mathcal{N}(r) \setminus \{0\}) \left[|\xi_n(\omega)| \leq \frac{1}{|a_n|r^n N^{2/3}(r)} \right]\right\}, \\ A_3 &= \left\{\omega: (\forall n \in \mathcal{N}_{\delta}(r) \setminus (\mathcal{N}(r) \cup \{0\})) \left[|\xi_n(\omega)| \leq \frac{1}{N^{2/3}(r)} \right]\right\}, \\ A_4 &= \left\{\omega: (\forall n \notin \mathcal{N}_{\delta}(r) \cup \mathcal{N}' \cup \{0\}) \left[|\xi_n(\omega)| \leq n \right]\right\}, \quad \delta = \frac{1}{2 \ln N(r)}. \end{aligned}$$

If Ω_1 occurs, then for $r \notin E$ we obtain

$$\begin{aligned}
& |\xi_0(\omega)a_0| - \left| \sum_{n=1}^{+\infty} \xi_n(\omega)a_n r^n \right| \geq 2eN^{1/3}(r) \exp\{2\sqrt{\ln N(r)}\} - \\
& - \sum_{n \in \mathcal{N}(r)} \frac{|a_n|r^n}{|a_n|r^n N^{2/3}(r)} - \sum_{n \in \mathcal{N}_\delta(r) \setminus \mathcal{N}(r)} \frac{|a_n|r^n}{N^{2/3}(r)} - \sum_{n \notin \mathcal{N}_\delta(r) \cup \mathcal{N}'} ne^{-n\delta} > \\
& > 2eN^{1/3}(r) \exp\{2\sqrt{\ln N(r)}\} - \sum_{n \in \mathcal{N}_\delta(r)} \frac{1}{N^{2/3}(r)} - \\
& - \int_1^{+\infty} xe^{-\delta x} dx > 2eN^{1/3}(r) \exp\{2\sqrt{\ln N(r)}\} - \\
& - N^{1/3}(r) - eN^{1/3}(r) \exp\{2\sqrt{\ln N(r)}\} - 8 \ln^2 N(r) > 0
\end{aligned}$$

as $r \rightarrow +\infty$, because

$$\int_1^{+\infty} xe^{-\delta x} dx = \frac{e^{-\delta}}{\delta^2}(\delta + 1) < \frac{2}{\delta^2} = 8 \ln^2 N(r).$$

So, we proved that first term dominates the sum of all the other terms inside $r\mathbb{D}$, i.e.

$$|\xi_0(\omega)a_0| > \left| \sum_{n=1}^{+\infty} \xi_n(\omega)a_n z^n \right|. \quad (6)$$

If Ω_1 occurs then the function $f(z, \omega)$ has no zeros inside $r\mathbb{D}$. Now we find a lower bound for the probability of the event Ω_1 .

$$\begin{aligned}
P(A_1) &= \exp\left\{-\frac{4e^2 N^{2/3}(r) \exp\{4\sqrt{\ln N(r)}\}}{|a_0|^2}\right\}, \\
P(A_2) &\geq \prod_{n \in \mathcal{N}(r)} \frac{1}{2|a_n|^2 r^{2n} N^{4/3}(r)} = \prod_{n \in \mathcal{N}(r)} \frac{1}{2|a_n|^2 r^{2n}} \times \\
&\times \exp\{-N(r) \ln(N^{4/3}(r))\} = \exp\left\{-s(r) - \frac{4}{3}N(r) \ln N(r) - N(r) \ln 2\right\}, \\
P(A_3) &\geq \prod_{n \in \mathcal{N}(re^\delta)} \frac{1}{2N^{4/3}(r)} \geq \exp\left\{-N(re^\delta) \ln(2N^{4/3}(r))\right\} \geq \\
&\geq \exp\left\{-eN(r) \exp\{2\sqrt{\ln N(r)}\} \ln(2N^{4/3}(r))\right\}, \\
P(A_4) &= P\{\omega: (\forall n \notin \mathcal{N}_\delta(r) \cup \mathcal{N}' \cup \{0\}) |\xi_n(\omega)| < n\} \geq \\
&\geq 1 - \sum_{n \notin \mathcal{N}_\delta(r) \cup \mathcal{N}' \cup \{0\}} e^{-n^2} > \frac{1}{2}, \quad r \rightarrow +\infty.
\end{aligned}$$

It follows from $\Omega_1 \subset \{\omega : n(r, \omega) = 0\}$ that for any $\varepsilon > 0$ and for every $r \in [r_0, +\infty) \setminus E$

$$\begin{aligned} p_0(r) &= \ln^- P\{\omega : n(r, \omega) = 0\} \leq \ln^- P(\Omega_1) = \sum_{n=1}^4 \ln^- P(A_n) \leq \\ &\leq \ln 2 + \frac{4e^2 N^{2/3}(r) \exp\{4\sqrt{\ln N(r)}\}}{|a_0|^2} + s(r) + 2N(r) \ln N(r) + N(r) \ln 2 + \\ &+ eN(r) \exp\{2\sqrt{N(r)}\} \ln(2N^{4/3}(r)) \leq s(r) + N(r) \exp\{(2 + \varepsilon)\sqrt{N(r)}\} \end{aligned}$$

□

A random entire function of the form

$$g(z, \omega) = \sum_{n=0}^{+\infty} e^{2\pi i \theta_n(\omega)} a_n z^n \quad (7)$$

where independent random variables $\theta_n(\omega)$ are uniformly distributed on $(0, 1)$, was considered in [14]. For such functions there were proved the following statements.

Theorem 4.2 ([12]). *Let $g(z, \omega)$ be a random entire function of the form (7). Then for $r > r_0$ we obtain*

$$N_g(r, \omega) \leq \frac{1}{2e} + \ln S_g(r),$$

where

$$N_g(r, \omega) = \frac{1}{2\pi} \int_0^{2\pi} \ln |g(re^{i\alpha}, \omega)| d\alpha - \ln |a_k|,$$

and $k = \min\{n \in \mathbb{Z}_+ : a_n \neq 0\}$.

Theorem 4.3 ([14]). *There exists an absolute constant $C > 0$ such that for a function $g(z, \omega)$ of the form (7) almost surely we have*

$$\ln S_g(r) \leq N_g(r, \omega) + C \ln N_g(r, \omega), \quad r_0(\omega) \leq r < +\infty. \quad (8)$$

Corollary 4.4. *Let $(\zeta_n(\omega))$ be a sequence of independent identically distributed random variables such that for any $n \in \mathbb{N}$ random variable $\arg \zeta_n(\omega)$ is uniformly distributed on $[-\pi, \pi)$ and $M|\xi_n| < +\infty$, $M(\frac{1}{|\xi_n|}) < +\infty$, $n \in \mathbb{Z}_+$. Then there exists an absolute constant $C > 0$ such that for every random function of the form $f(z, \omega) = \sum_{n=0}^{+\infty} \zeta_n(\omega) a_n z^n$ we get almost surely*

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\alpha}, \omega)| d\alpha - \ln |a_k \zeta_k(\omega)| \geq \ln S_f(r, \omega) - (C + 1) \ln \ln S_f(r, \omega)$$

for $r_0(\omega) \leq r < +\infty$ and $k = \min\{n \in \mathbb{Z}_+ : a_n \neq 0\}$.

Since random variables $\arg \xi_n(\omega)$ (here $\xi_n(\omega) \in \mathcal{N}_{\mathbb{C}}(0, 1)$) are also uniformly distributed on $[-\pi, \pi)$, we have the following statement for the functions of the form (3).

Corollary 4.5. *There exists an absolute constant $C > 0$ such that for the functions of the form (3) we get almost surely*

$$\frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta}, \omega)| d\theta - \ln |a_k \xi_k(\omega)| \geq \ln S_f(r, \omega) - (C + 1) \ln \ln S_f(r, \omega)$$

for $r_0(\omega) \leq r < +\infty$ and $k = \min\{n \in \mathbb{Z}_+ : a_n \neq 0\}$.

Proof of corollary 4.4. It follows from Theorem 4.2 that $\ln N_g(r, \omega) \leq 1 + \ln \ln S_g(r)$ and by Theorem 4.3 we have almost surely

$$N_g(r, \omega) \geq \ln S_g(r) - C \ln N_g(r, \omega) \geq \ln S_g(r) - (C + 1) \ln \ln S_g(r),$$

for $r_0(\omega) \leq r < +\infty$. Therefore

$$P\{\omega : (\exists r_0(\omega))(\forall r > r_0(\omega)) [N_g(r, \omega) \geq \ln S_g(r) - (C + 1) \ln \ln S_g(r)]\} = 1.$$

Consider a random function

$$f(z, \omega_1, \omega_2) = \sum_{n=0}^{+\infty} \varepsilon_n(\omega_1) \eta_n(\omega_2) a_n z^n,$$

where $\varepsilon_n(\omega_1) = e^{i\theta_n(\omega_1)}$, $\theta_n(\omega_1)$ and $\arg \eta_n(\omega_2)$ are uniformly distributed on $[-\pi, \pi)$. Also both sequences $\{\varepsilon_n(\omega_1)\}$, $\{\eta_n(\omega_2)\}$ are sequences of independent random variables defined on the Steinhaus probability spaces $(\Omega_1, \mathcal{A}_1, P_1)$ and $(\Omega_2, \mathcal{A}_2, P_2)$, respectively. Define

$$A_f = \{(\omega_1, \omega_2) : (\exists r_0(\omega_1, \omega_2))(\forall r > r_0(\omega_1, \omega_2)) [N_f(r, \omega_1, \omega_2) \geq \ln S_f(r, \omega_2) - (C + 1) \ln \ln S_f(r, \omega_2)]\},$$

where

$$S_f^2(r, \omega_2) = \sum_{n=0}^{+\infty} |\varepsilon_n(\omega_1)|^2 |\eta_n(\omega_2)|^2 |a_n|^2 r^{2n} = \sum_{n=0}^{+\infty} |\eta_n(\omega_2)|^2 |a_n|^2 r^{2n}.$$

Consider the events

$$F = \{\omega_2 : (\forall n \in \mathbb{N}) [\eta_n(\omega_2) \neq 0]\}, \quad H = \left\{ \omega_2 : \overline{\lim}_{n \rightarrow +\infty} \sqrt[n]{|a_n| |\eta_n(\omega_2)|} = 0 \right\}.$$

Then by Lemma 3.5 $P_2(H) = 1$. Since $M(\frac{1}{|\xi_n|}) < +\infty$, the probability of the event F

$$1 \geq P_2(F) \geq 1 - \sum_{n=0}^{+\infty} P_2\{\omega_2 : \eta_n(\omega_2) = 0\} = 1.$$

Denote $G = F \cap H$. So, $P_2(G) = 1$. Then for fixed $\omega_2^0 \in G$

$$P_1(A_f(\omega_2^0)) := P_1\{\omega_1 : (\exists r_0(\omega_1, \omega_2^0))(\forall r > r_0(\omega_1, \omega_2^0)) \\ [N_f(r, \omega_1, \omega_2^0) \geq \ln S_f(r, \omega_2^0) - (C+1) \ln \ln S_f(r, \omega_2^0)]\} = 1.$$

Let P be a direct product of the probability measures P_1 and P_2 defined on $(\Omega_1 \times \Omega_2, \mathcal{A}_1 \times \mathcal{A}_2, P_1 \times P_2)$. Here $\mathcal{A}_1 \times \mathcal{A}_2$ is the σ -algebra, which contains all $A_1 \times A_2$ such that $A_1 \in \mathcal{A}_1$ and $A_2 \in \mathcal{A}_2$. By Fybin's theorem

$$P(A_f) = \int_{\Omega_2} \left(\int_{A_f(\omega_2)} dP_1(\omega_1) \right) dP_2(\omega_2) \geq \int_G \left(\int_{A_f(\omega_2)} dP_1(\omega_1) \right) dP_2(\omega_2) = \\ = \int_G dP_2(\omega_2) = P_2(G) = 1.$$

Suppose that there exists a set $B_1 \in \Omega_1$ such that $P_1(B_1) > 0$ and for all fixed $\omega_1^0 \in B_1$

$$P_2(A_f(\omega_1^0)) = P_2\{\omega_2 : (\exists r_0(\omega_1^0, \omega_2))(\forall r > r_0(\omega_1^0, \omega_2)) \\ [N_f(r, \omega_1^0, \omega_2) \geq \ln S_f(r, \omega_2) - (C+1) \ln \ln S_f(r, \omega_2)]\} < 1.$$

Then

$$P(A_f) = \int_{B_1} \left(\int_{A_f(\omega_1)} dP_2(\omega_2) \right) dP_1(\omega_1) + \int_{\Omega_2 \setminus B_1} \left(\int_{A_f(\omega_1)} dP_2(\omega_2) \right) dP_1(\omega_1) < \\ < \int_{B_1} dP_1(\omega_1) + \int_{\Omega_2 \setminus B_1} dP_1(\omega_1) = 1.$$

Contradiction.

Therefore, almost surely by ω_1 we have $P_2(A_f(\omega_1)) = 1$, i.e. there exists $B_2 \in \Omega_1 : P_1(B_2) = 1$ and for all $\omega_1^0 \in B_2$ we get

$$P_2\{\omega_2 : (\exists r_0(\omega_1^0, \omega_2))(\forall r > r_0(\omega_1^0, \omega_2)) \\ [N_f(r, \omega_1^0, \omega_2) \geq \ln S_f(r, \omega_2) - (C+1) \ln \ln S_f(r, \omega_2)]\} = 1.$$

We fix $\omega_1^0 \in B_2$. Let $\varepsilon(\omega_1^0) = \varepsilon^*$, $n \in \mathbb{Z}_+$. Then for the random entire function

$$h(z, \omega) = \sum_{n=0}^{+\infty} \varepsilon_n^* \eta_n(\omega) a_n z^n,$$

we obtain

$$P\{\omega : (\exists r_0(\omega))(\forall r > r_0(\omega)) [N_h(r, \omega) \geq \ln S_h(r, \omega) - (C+1) \ln \ln S_h(r, \omega)]\} = 1.$$

Remark that the sequence $\{\varepsilon_n^* \eta_n(\omega)\}$ is the sequence of independent random variables and sequences $\{\varepsilon_n^* \eta_n(\omega)\}$, $\{\eta_n(\omega)\}$ are similar. It remains to denote $\zeta_n(\omega) = \varepsilon_n^* \eta_n(\omega)$. \square

Theorem 4.6. *Let f be a random entire function of the form (3) such that $a_0 \neq 0$. Then there exists $r_0 > 0$ such that for all $r \in (r_0; +\infty)$ we get*

$$p_0(r) \geq s(r) + N(r) \ln N(r) - 4N(r).$$

Proof of Theorem 4.6. By Jensen's formula we get almost surely

$$\begin{aligned} 0 &= \int_0^r \frac{n(t, \omega)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta}, \omega)| d\theta - \ln |a_0 \xi_0(\omega)|, \\ \ln |a_0 \xi_0(\omega)| &= \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta}, \omega)| d\theta. \end{aligned}$$

Therefore,

$$P\{\omega: n(r, \omega) = 0\} \leq P\left\{\omega: \ln |a_0 \xi_0(\omega)| = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta}, \omega)| d\theta\right\}.$$

Let $G_1 = \{\omega: \ln |a_0 \xi_0(\omega)| \geq \ln \gamma(\omega)\}$, $G_2 = \left\{\omega: \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta}, \omega)| d\theta \leq \ln \gamma(\omega)\right\}$, where $\gamma(\omega) > 1$. Then

$$\begin{aligned} \overline{G_1} \cap \overline{G_2} &= \left\{\omega: \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta}, \omega)| d\theta > \ln \gamma(\omega) > \ln |a_0 \xi_0(\omega)|\right\}, \\ \overline{G_1} \cap \overline{G_2} &\subset \left\{\omega: \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta}, \omega)| d\theta \neq \ln |a_0 \xi_0(\omega)|\right\}, \\ G_1 \cup G_2 &= \overline{\overline{G_1} \cap \overline{G_2}} \supset \left\{\omega: \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta}, \omega)| d\theta = \ln |a_0 \xi_0(\omega)|\right\}. \end{aligned}$$

So,

$$P\{\omega: n(r, \omega) = 0\} \leq P(G_1 \cup G_2) \leq P(G_1) + P(G_2). \quad (9)$$

We define

$$\begin{aligned} A &= \left\{\omega: (\exists r_0(\omega))(\forall r > r_0(\omega)) \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta}, \omega)| d\theta \geq \right. \\ &\quad \left. \geq \ln S_f(r, \omega) - (C + 1) \ln \ln S_f(r, \omega) + \ln |a_0 \xi_0(\omega)|\right\}. \end{aligned}$$

By Corollary 4.5 we obtain that $P(A) = 1$. Put $\gamma(r, \omega) = C_1 \cdot |a_0| \cdot |\xi_0(\omega)|$, $C_1 > 1$. Then we may calculate the probability of the event G_1

$$P(G_1) = P\{\omega: \ln |a_0 \xi_0(\omega)| \geq \ln C_1 + \ln |a_0 \xi_0(\omega)|\} = P\{\omega: \ln C_1 \leq 0\} = 0$$

and estimate the probability of the event G_2 as $r \rightarrow +\infty$

$$\begin{aligned}
P(G_2) &= P(G_2 \cap A) + P(G_2 \cap \overline{A}) \leq P(G_2 \cap A) + P(\overline{A}) = P(G_2 \cap A) = \\
&= P\left\{\omega: (\exists r_0(\omega))(\forall r > r_0(\omega)) \left[\ln S_f(r, \omega) - (C+1) \ln \ln S_f(r, \omega) + \right. \right. \\
&\quad \left. \left. + \ln |a_0 \xi_0(\omega)| \leq \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta}, \omega)| d\theta \leq \ln \gamma(r, \omega) \right] \right\} = \\
&= P\left\{\omega: (\exists r_0(\omega))(\forall r > r_0(\omega)) \left[\ln S_f(r, \omega) - (C+1) \ln \ln S_f(r, \omega) + \ln |a_0 \xi_0(\omega)| \leq \ln C_1 + \ln |a_0 \xi_0(\omega)| \right] \right\} = \\
&= P\left\{\omega: (\exists r_0(\omega))(\forall r > r_0(\omega)) \left[\ln S_f(r, \omega) - (C+1) \ln \ln S_f(r, \omega) \leq \ln C_1 \right] \right\} \leq \\
&\leq P\left\{\omega: (\exists r_0(\omega))(\forall r > r_0(\omega)) \left[\ln S_f(r, \omega) \leq 2 \ln C_1 \right] \right\} = \\
&= P\left\{\omega: (\exists r_0(\omega))(\forall r > r_0(\omega)) \left[S_f(r, \omega) \leq C_1^2 \right] \right\} = P\left\{\omega: S_f(r, \omega) \leq C_1^2 \right\} \leq \\
&\leq P\left\{\omega: \sum_{n \in \mathcal{N}(r)} |\xi_n(\omega)|^2 |a_n|^2 r^{2n} \leq C_1^4 \right\}. \tag{10}
\end{aligned}$$

The function of distribution of the random variable $|\xi_n(\omega)|$

$$\begin{aligned}
F_{|\xi_n|}(x) &= 1 - \exp\{-x^2\}, \quad F_{|\xi_n|^2}(x) = F_{|\xi_n|}(\sqrt{x}) = 1 - \exp\{-x\}, \\
F_{|\xi_n|^2 |a_n|^2 r^{2n}}(x) &= F_{|\xi_n|^2}\left(\frac{x}{|a_n|^2 r^{2n}}\right) = 1 - \exp\left\{-\frac{x}{|a_n|^2 r^{2n}}\right\}
\end{aligned}$$

for $n \notin \mathcal{N}'$ and $x \in \mathbb{R}_+$. Then for the random vector $\eta(\omega) = (|\xi_1(\omega)|a_1 r^{j_1}, \dots, \xi_{j_k}(\omega)|a_{j_k} r^{j_k}), j_k \in \mathcal{N}(r)$, the density function

$$p_\eta(x) = \begin{cases} \prod_{n \in \mathcal{N}(r)} \frac{1}{|a_n|^2 r^{2n}} \exp\left\{-\frac{x_n}{|a_n|^2 r^{2n}}\right\}, & x \in \mathbb{R}_+^{\mathcal{N}(r)}, \\ 0, & x \notin \mathbb{R}_+^{\mathcal{N}(r)}. \end{cases}$$

So,

$$\begin{aligned}
P\left\{\omega: \sum_{n \in \mathcal{N}(r)} |\xi_n(\omega)|^2 |a_n|^2 r^{2n} \leq C_1^4\right\} &= P\{\omega: \eta(\omega) \in B(r)\} = \\
&= \prod_{n \in \mathcal{N}(r)} \frac{1}{|a_n|^2 r^{2n}} \cdot \int \cdots \int_{B(r)} \prod_{n \in \mathcal{N}(r)} \exp\left\{-\frac{x_n}{|a_n|^2 r^{2n}}\right\} dx_1 \cdots dx_k \leq \\
&\leq \exp(-s(r)) \cdot \text{vol}_{\mathbb{R}^{\mathcal{N}(r)}} B(r), \tag{11}
\end{aligned}$$

where

$$B(r) = \left\{ x \in \mathbb{R}_+^{\mathcal{N}(r)}: \sum_{n \in \mathcal{N}(r)} x_n \leq C_1^4 \right\}.$$

For $C > 0$ by elementary calculation we get

$$\text{vol}_{\mathbb{R}^n} \left\{ x \in \mathbb{R}_+^n : \sum_{i=1}^n x_i \leq C \right\} = \frac{C^n}{n!}.$$

From this equality and Stirling's formula

$$n! = \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \cdot \exp \left\{ -\frac{\theta_n}{12n} \right\}, \quad \theta_n \in [0, 1], \quad n \in \mathbb{N},$$

it follows that the volume of the set $B(r)$

$$\begin{aligned} \ln \left(\text{vol}_{\mathbb{R}^{N(r)}} B(r) \right) &\leq -\frac{1}{2} \ln(2\pi) - \frac{1}{2} \ln N(r) - N(r) \ln N(r) + \frac{1}{12N(r)} + \\ &+ N(r) + 4N(r) \ln C_1 \leq -N(r)(\ln N(r) - 1 - 4 \ln C_1). \end{aligned}$$

Let us choose $C_1 = 2$. From (11) it follows $p_0(r) \geq s(r) + N(r) \ln N(r) - 4N(r)$. \square

Using Lemma 3.3 from Theorems 4.1 and 4.6 we deduce such a statement.

Theorem 4.7. *Let $\varepsilon > 0$, and f be a random entire function of the form (3) such that $a_0 \neq 0$. Then there exist $r_0 > 0$ and the set $E \subset (1; +\infty)$ of finite logarithmic measure such that for all $r \in (r_0; +\infty) \setminus E$ we obtain*

$$(1 - \varepsilon)N(r) \ln N(r) \leq p_0(r) - s(r) \leq N(r) \exp\{(2 + \varepsilon)\sqrt{\ln N(r)}\},$$

in particular,

$$0 \leq \lim_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln(p_0(r) - s(r))}{\ln s(r)}, \quad \overline{\lim}_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln(p_0(r) - s(r))}{\ln s(r)} \leq \frac{1}{2} \quad (12)$$

and

$$\lim_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln(p_0(r) - s(r))}{\ln N(r)} = 1.$$

5 Examples on sharpness of inequalities (12)

Theorem 5.1. *There exist a random entire function of form (3) for which $a_0 \neq 0$ and a set E of finite logarithmic measure such that*

$$\lim_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln(p_0(r) - s(r))}{\ln s(r)} = \frac{1}{2}.$$

Proof. Consider the entire function

$$f(z) = 1 + \sum_{n=1}^{+\infty} \frac{z^n}{\left(\frac{n}{2}\right)^{\frac{n}{2}}}.$$

For this function and $r \in (r_0(\omega); +\infty) \setminus E$ we have

$$\frac{\sqrt{s(r)}}{\ln^3 s(r)} < N(r) < \sqrt{s(r)} \exp\{(1 + \varepsilon)\sqrt{\ln s(r)}\}, \quad \lim_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln N(r)}{\ln s(r)} = \frac{1}{2}.$$

By Theorem 4.7 we have for $r \in (r_0; +\infty) \setminus E$

$$\begin{aligned} \frac{-\ln 2 + \ln N(r) + \ln \ln N(r)}{\ln s(r)} &\leq \frac{\ln(p_0(r) - s(r))}{\ln s(r)} \leq \frac{\ln N(r) + 3\sqrt{\ln N(r)}}{\ln s(r)}, \\ \lim_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln(p_0(r) - s(r))}{\ln s(r)} &= \lim_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln N(r)}{\ln s(r)} = \frac{1}{2}. \end{aligned}$$

□

Theorem 5.2. *There exist a random entire function of form (3) for which $a_0 \neq 0$ and a set E of finite logarithmic measure such that*

$$\lim_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln(p_0(r) - s(r))}{\ln s(r)} = 0.$$

Proof. Consider the entire functions

$$f(z) = 1 + \sum_{n=1}^{+\infty} \frac{z^n}{\left(\frac{n}{2}\right)^{\frac{n}{2}}}, \quad g(z) = 1 + \sum_{n \in \mathcal{N}^*} \frac{z^n}{\left(\frac{n}{2}\right)^{\frac{n}{2}}},$$

where $\mathcal{N}^* = \{n: n = [e^k] + 1 \text{ for some } k \in \mathbb{Z}_+\}$. Here $[e^k]$ means the integral part of the real number e^k . We denote

$$\begin{aligned} \mathcal{N}_f(r) &= \{n \in \mathbb{Z}_+: \ln(|a_n|r^n) > 0\} \setminus \{0\}, \quad \mathcal{N}_g(r) = \{n \in \mathcal{N}^*: \ln(|a_n|r^n) > 0\}, \\ s_f(r) &= 2 \sum_{n \in \mathcal{N}_f(r)} \ln(|a_n|r^n), \quad s_g(r) = 2 \sum_{n \in \mathcal{N}_g(r)} \ln(|a_n|r^n), \quad a_n = \left(\frac{n}{2}\right)^{-\frac{n}{2}}, \quad n \in \mathbb{N}. \end{aligned}$$

Remark that the sequence $\{(n/2)^{-n/2}\}$ is log-concave and

$$\mathcal{N}_f(r) = \{1, \dots, N_f(r)\}.$$

Then by the definition of $N_g(r)$ we get $N_g(r) \leq 2 \ln N_f(r)$, $r \rightarrow +\infty$. For $r \in (r_0; +\infty) \setminus E$ we obtain (see [22])

$$N_g(r) \leq 2 \ln N_f(r) \leq 2 \ln(\ln \mu_f(r) \ln^2(\ln \mu_f(r))) < 4 \ln \ln \mu_f(r).$$

Remark that $\min\{n \in \mathcal{N}' : n > \nu_g(r)\} \leq [e\nu_g(r)] + 1 < (e+1)\ln \nu_g(r)$. Let us fix $r > 0$. Consider the function $y(t) = \ln(a(t)r^t) = -\frac{t}{2}\ln(\frac{t}{2}) + t\ln r$, for which $a(n) = a_n$. The graph of the function $y(t)$ passes through the points $(0; 0)$ and $(\nu_g(r), \ln \mu_g(r))$. It follows from log-concavity of the function $y(t)$ that the point $(\nu_f(r), \ln \mu_f(r))$ belongs to the triangle with the vertices $(\nu_g(r), \ln \mu_g(r))$, $((e+1)\nu_g(r), \ln \mu_g(r))$ and $((e+1)\nu_g(r), (e+1)\ln \mu_g(r))$. Then

$$\ln \mu_f(r) \leq (e+1)\ln \mu_g(r), \quad s_g(r) \geq 2\ln \mu_g(r) \geq \frac{2}{e+1}\ln \mu_f(r).$$

For the function $g(z)$ and $r \in (r_0; +\infty) \setminus E$ we get

$$0 \leq \lim_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln(p_0(r) - s_g(r))}{\ln s_g(r)} = \lim_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln N_g(r)}{\ln s_g(r)} \leq \lim_{\substack{r \rightarrow +\infty \\ r \notin E}} \frac{\ln(4\ln \ln \mu_f(r))}{\ln(\frac{2}{e+1}\ln \mu_f(r))} = 0.$$

□

6 Zeros distribution of non-gaussian entire functions

One can ask *what happens when random variables $\xi_n(\omega)$ in (3) have not gaussian distribution([21])?* The following result shows that the situation may be very different.

Theorem 6.1. *Let $f(z, \omega) = \sum_{n=0}^{+\infty} \zeta_n(\omega) a_n z^n$, $a_0 \neq 0$, with a sequence of independent identically distributed random variables $(\zeta_n(\omega))_{n \in \mathbb{Z}_+}$ such that*

- 1) $(\arg \zeta_n(\omega))_{n \in \mathbb{Z}_+}$ are uniformly distributed on $[-\pi, \pi)$;
- 2) $M|\zeta_n| < +\infty$ and $M(\frac{1}{|\zeta_n|}) < +\infty$, $n \in \mathbb{Z}_+$;
- 3) there exists $\varepsilon > 0$ such that for any $n \in \mathbb{Z}_+$ we have $P\{\omega : |\zeta_n(\omega)| < \varepsilon\} = 0$.

Then there exists $r_0 > 0$ such that for all $r > r_0$ we get

$$P\{\omega : n(f, \omega) = 0\} = 0.$$

Proof. From inequality (10) we get for some constant $C_3 > 0$

$$\begin{aligned} 0 \leq P\{\omega : n(f, \omega) = 0\} &\leq P\left\{\omega : \sum_{n \in \mathcal{N}(r)} |\zeta_n(\omega)|^2 |a_n|^2 r^{2n} \leq C_3\right\} \leq \\ &\leq P\left\{\omega : \sum_{n \in \mathcal{N}(r)} \varepsilon^2 |a_n|^2 r^{2n} \leq C_3\right\} = 0 \quad (r \rightarrow +\infty). \end{aligned}$$

□

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